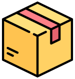
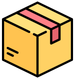
Strategic Private Exploration (with Destructive Ties)

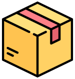
Nawaaz Khalfan
Monash University

AETW 2026

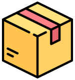
Some boxes . . .



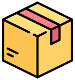




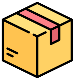
A strategic exploration game



Box A



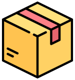
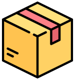
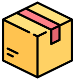
Box B



Box C

A strategic exploration game

A one dollar prize is hidden in one of three boxes with equal probability. You and I will compete to be the first to find the prize; the winner keeps the dollar! We may choose the order in which to search the boxes, but must conduct our search in *private*.

		
Box A	Box B	Box C
$\pi_A = 1$	$\pi_B = 1$	$\pi_C = 1$
$p_A = \frac{1}{3}$	$p_B = \frac{1}{3}$	$p_C = \frac{1}{3}$

Economics

1. **The paparazzi problem** - Fershtman and Rubinstein (1997): reporters race to get the *scoop* on a celebrity staying at one of many hotels in the city

2. **A ranking duel** - Immorlica et al. (2011): search engines contend to provide the correct link to an internet surfer

3. **Market servicing**: entrepreneurs compete to service a missing market

A general exploration game

- There are n boxes available, and box i has prize π_i with probability p_i.

A general exploration game

- There are n boxes available, and box i has prize π_i with probability p_i.
- Two players, player 1 and player 2, simultaneously choose the sequence in which they explore the boxes. There are two ways of interpreting this choice as "exploration":
  1. exploration occurs in private and the distribution of the prizes is such that, after exploring each box the relative incentives for searching each of the unexplored boxes remains the same, or
  2. once the exploration sequence is chosen it cannot be altered.

A general exploration game

- There are n boxes available, and box i has prize π_i with probability p_i.
- Two players, player 1 and player 2, simultaneously choose the sequence in which they explore the boxes. There are two ways of interpreting this choice as "exploration":
  1. exploration occurs in private and the distribution of the prizes is such that, after exploring each box the relative incentives for searching each of the unexplored boxes remains the same, or
  2. once the exploration sequence is chosen it cannot be altered.
- Each player can explore one box per unit of time, independent of how profitable the box is.

A general exploration game

- There are n boxes available, and box i has prize π_i with probability p_i.
- Two players, player 1 and player 2, simultaneously choose the sequence in which they explore the boxes. There are two ways of interpreting this choice as "exploration":
  1. exploration occurs in private and the distribution of the prizes is such that, after exploring each box the relative incentives for searching each of the unexplored boxes remains the same, or
  2. once the exploration sequence is chosen it cannot be altered.
- Each player can explore one box per unit of time, independent of how profitable the box is.
- The marginal cost of exploring box i is zero. That is, we are not concerned with the stopping problem, only the order in which they search.

A general exploration game

- There are n boxes available, and box i has prize π_i with probability p_i.
- Two players, player 1 and player 2, simultaneously choose the sequence in which they explore the boxes. There are two ways of interpreting this choice as "exploration":
  1. exploration occurs in private and the distribution of the prizes is such that, after exploring each box the relative incentives for searching each of the unexplored boxes remains the same, or
  2. once the exploration sequence is chosen it cannot be altered.
- Each player can explore one box per unit of time, independent of how profitable the box is.
- The marginal cost of exploring box i is zero. That is, we are not concerned with the stopping problem, only the order in which they search.
- The first player to find a prize gets to keep it. If the prize is found at the same time, then it will be "split" in some predetermined fashion ...

Ties

If both players explore box i in the same unit of time then we say that the players have **tied**. Denote the share of the prize allocated to player 1 and player 2 as $s_1, s_2 \in [0, 1]$.

Ties

If both players explore box i in the same unit of time then we say that the players have **tied**. Denote the share of the prize allocated to player 1 and player 2 as $s_1, s_2 \in [0, 1]$. There are three cases:

- **Split ties** where $s_1 + s_2 = 1$. For example, there may be some exogenous random variable determines the split.
- **Destructive ties** where $s_1 + s_2 < 1$. For example, there may be a further round of costly competition.
- **Productive ties** where $s_1 + s_2 > 1$. For example, there may be some notion of joint credit.

Ties

If both players explore box i in the same unit of time then we say that the players have **tied**. Denote the share of the prize allocated to player 1 and player 2 as $s_1, s_2 \in [0, 1]$. There are three cases:

- **Split ties** where $s_1 + s_2 = 1$. For example, there may be some exogenous random variable determines the split.
- **Destructive ties** where $s_1 + s_2 < 1$. For example, there may be a further round of costly competition.
- **Productive ties** where $s_1 + s_2 > 1$. For example, there may be some notion of joint credit.

Today, I will tell you about:

1. completely destructive ties where $s_1 = s_2 = 0$, and
2. show, as an example, how you can generalize to even split ties $s_1 = s_2 = \frac{1}{2}$.

Strategies

A pure strategy in this game is a permutation of the set $N := \{1, 2, \ldots, n\}$, and a mixed strategy is a convex combination of these permutations. Given the assumptions about time and search speed, having i in the jth position can be thought of as searching box i at *time step* j.

Strategies

A pure strategy in this game is a permutation of the set $N := \{1, 2, \ldots, n\}$, and a mixed strategy is a convex combination of these permutations. Given the assumptions about time and search speed, having i in the jth position can be thought of as searching box i at *time step* j.

Let $P_1 \in \Delta\mathcal{P}$ denote the strategy for player 1 where $\mathcal{P}$ is the set of all permutations of N, and P_2 for player 2.

A simple example

Consider a uniform 3 box environment where $\pi_A = \pi_B = \pi_C = 6$ and $p_A = p_B = p_C = \frac{1}{3}$.

A simple example

Consider a uniform 3 box environment where $\pi_A = \pi_B = \pi_C = 6$ and $p_A = p_B = p_C = \frac{1}{3}$. This gives us the following payoff matrix:

	ABC	ACB	BAC	BCA	CAB	CBA
ABC	0,0	2,2	2,2	**2,4**	**4,2**	2,2
ACB	2,2	0,0	**4,2**	2,2	2,2	**2,4**
BAC	2,2	**2,4**	0,0	2,2	2,2	**4,2**
BCA	**4,2**	2,2	2,2	0,0	**2,4**	2,2
CAB	**2,4**	2,2	2,2	**4,2**	0,0	2,2
CBA	2,2	**4,2**	**2,4**	2,2	2,2	0,0

A simple example

Consider a uniform 3 box environment where $\pi_A = \pi_B = \pi_C = 6$ and $p_A = p_B = p_C = \frac{1}{3}$. This gives us the following payoff matrix:

	ABC	ACB	BAC	BCA	CAB	CBA
ABC	0,0	2,2	2,2	**2,4**	**4,2**	2,2
ACB	2,2	0,0	**4,2**	2,2	2,2	**2,4**
BAC	2,2	**2,4**	0,0	2,2	2,2	**4,2**
BCA	**4,2**	2,2	2,2	0,0	**2,4**	2,2
CAB	**2,4**	2,2	2,2	**4,2**	0,0	2,2
CBA	2,2	**4,2**	**2,4**	2,2	2,2	0,0

The (strategically) unique Nash equilibrium of this game is:

$$P_1^* = P_2^* = \frac{1}{3} \circ ABC + \frac{1}{3} \circ BCA + \frac{1}{3} \circ CAB$$

This gives each player an expected value of 2.0.

An irregular example

Consider a non-uniform 3 box environment where $\pi_A = 6$, $\pi_B = 3$ and $\pi_c = 2$, all with probability one.

An irregular example

Consider a non-uniform 3 box environment where $\pi_A = 6$, $\pi_B = 3$ and $\pi_c = 2$, all with probability one. This gives us the following payoff matrix:

	ABC	ACB	BAC	BCA	CAB	CBA
ABC	0,0	3,2	6,3	**6,5**	**9,2**	6,2
ACB	2,3	0,0	**8,3**	6,3	6,2	**6,5**
BAC	3,6	**3,8**	0,0	6,2	3,2	**9,2**
BCA	**5,6**	3,6	2,6	0,0	**3,8**	3,2
CAB	**2,9**	2,6	2,3	**8,3**	0,0	6,3
CBA	2,6	**5,6**	**2,9**	2,3	3,6	0,0

An irregular example: levelling equilibrium

The discrete levelling equilibrium is the following:

$$P_1^* = P_2^* = \frac{1}{3} \circ ABC + \frac{1}{3} \circ ACB + \frac{1}{6} \circ BAC + \frac{1}{6} \circ BCA$$

An irregular example: levelling equilibrium

The discrete levelling equilibrium is the following:

$$P_1^* = P_2^* = \frac{1}{3} \circ ABC + \frac{1}{3} \circ ACB + \frac{1}{6} \circ BAC + \frac{1}{6} \circ BCA$$

	ABC	ACB	BAC	BCA	CAB	CBA
ABC	0,0	3,2	6,3	**6,5**	**9,2**	6,2
ACB	2,3	0,0	**8,3**	6,3	6,2	**6,5**
BAC	3,6	**3,8**	0,0	6,2	3,2	**9,2**
BCA	**5,6**	3,6	2,6	0,0	**3,8**	3,2
CAB	**2,9**	2,6	2,3	**8,3**	0,0	6,3
CBA	2,6	**5,6**	**2,9**	2,3	3,6	0,0

This gives each player an expected value of 3.0.

A new example: asymmetric equilibrium

But consider the following Pareto dominant equilibrium in asymmetric strategies:

$$P_1^* = \frac{2}{3} \circ ABC + \frac{1}{3} \circ BAC,$$
$$P_2^* = \frac{2}{3} \circ ACB + \frac{1}{3} \circ BCA$$

A new example: asymmetric equilibrium

But consider the following Pareto dominant equilibrium in asymmetric strategies:

$$P_1^* = \frac{2}{3} \circ ABC + \frac{1}{3} \circ BAC,$$
$$P_2^* = \frac{2}{3} \circ ACB + \frac{1}{3} \circ BCA$$

	ABC	ACB	BAC	BCA	CAB	CBA
ABC	0,0	3,2	6,3	**6,5**	**9,2**	6,2
ACB	2,3	0,0	**8,3**	6,3	6,2	**6,5**
BAC	3,6	**3,8**	0,0	6,2	3,2	**9,2**
BCA	**5,6**	3,6	2,6	0,0	**3,8**	3,2
CAB	**2,9**	2,6	2,3	**8,3**	0,0	6,3
CBA	2,6	**5,6**	**2,9**	2,3	3,6	0,0

This gives each player an expected value of 4.0.

Payoffs

Let x_{ij} denote the induced probability that player 1 chooses to search box i in the jth position given P_1, and y_{ij} be the corresponding probability for player 2.

Payoffs

Let x_{ij} denote the induced probability that player 1 chooses to search box i in the jth position given P_1, and y_{ij} be the corresponding probability for player 2.

$$x_{ij}$$

Payoffs

Let x_{ij} denote the induced probability that player 1 chooses to search box i in the jth position given P_1, and y_{ij} be the corresponding probability for player 2.

$$\pi_i x_{ij}$$

Payoffs

Let x_{ij} denote the induced probability that player 1 chooses to search box i in the jth position given P_1, and y_{ij} be the corresponding probability for player 2.

$$p_i \pi_i x_{ij}$$

Payoffs

Let x_{ij} denote the induced probability that player 1 chooses to search box i in the jth position given P_1, and y_{ij} be the corresponding probability for player 2.

$$p_i \pi_i x_{ij} \left[\sum_{k>j} y_{ik} + \frac{1}{2} y_{ij} \right]$$

Payoffs

Let x_{ij} denote the induced probability that player 1 chooses to search box i in the jth position given P_1, and y_{ij} be the corresponding probability for player 2.

$$\sum_i \sum_j p_i \pi_i x_{ij} \left[\sum_{k>j} y_{ik} + \frac{1}{2} y_{ij} \right]$$

Payoffs

Let x_{ij} denote the induced probability that player 1 chooses to search box i in the jth position given P_1, and y_{ij} be the corresponding probability for player 2.

$$v_1(x,y) = \sum_i \sum_j p_i \pi_i x_{ij} \left[\sum_{k>j} y_{ik} + \frac{1}{2} y_{ij} \right]$$

Payoffs

Let x_{ij} denote the induced probability that player 1 chooses to search box i in the jth position given P_1, and y_{ij} be the corresponding probability for player 2.

$$v_1(x, y) = \sum_i \sum_j p_i \pi_i x_{ij} \left[\sum_{k>j} y_{ik} + \frac{1}{2} y_{ij} \right]$$

Claim: the expected value of the box is all that matters. From here on, let $r_i = p_i \pi_i$, and order the index of boxes from highest expected value to lowest.

Birkhoff–von Neumann

A doubly stochastic (DS) matrix is a square matrix of non-negative real numbers whose rows and columns sum to 1. That is, if z is a DS matrix then:

$$\sum_i z_{ij} = 1 \ \forall j \text{ , and } \sum_j z_{ij} = 1 \ \forall i$$

Birkhoff–von Neumann

A doubly stochastic (DS) matrix is a square matrix of non-negative real numbers whose rows and columns sum to 1. That is, if z is a DS matrix then:

$$\sum_i z_{ij} = 1 \ \forall j \text{ , and } \sum_j z_{ij} = 1 \ \forall i$$

This means that x, the matrix who's entries are given by x_{ij} and derived from P_1, is a DS matrix.

Birkhoff–von Neumann

A doubly stochastic (DS) matrix is a square matrix of non-negative real numbers whose rows and columns sum to 1. That is, if z is a DS matrix then:

$$\sum_i z_{ij} = 1 \ \forall j \text{ , and } \sum_j z_{ij} = 1 \ \forall i$$

This means that x, the matrix who's entries are given by x_{ij} and derived from P_1, is a DS matrix. Further, the **Birkhoff-von Nuemann theorem** guarantees that *any* DS matrix can be represented as a convex combination of permutation matrices.

Birkhoff–von Neumann

A doubly stochastic (DS) matrix is a square matrix of non-negative real numbers whose rows and columns sum to 1. That is, if z is a DS matrix then:

$$\sum_i z_{ij} = 1 \ \forall j \text{ , and } \sum_j z_{ij} = 1 \ \forall i$$

This means that x, the matrix who's entries are given by x_{ij} and derived from P_1, is a DS matrix. Further, the **Birkhoff-von Nuemann theorem** guarantees that *any* DS matrix can be represented as a convex combination of permutation matrices.

While this decomposition may not be a unique, the values for x_{ij} remain the same regardless of the combination of permutations. So it is without loss to use x and the analogous DS matrix y as our decision variables.

Equilibrium

As such, the problem for player 1, given y, is the following:

$$\max_x v_1(x, y) \text{ s.t. } \sum_i x_{ij} = 1 \ \forall j \text{ , and } \sum_j x_{ij} = 1 \ \forall i \tag{1}$$

where:

$$v_1(x, y) = \sum_i \sum_j r_i x_{ij} \left[\sum_{k>j} y_{ik} + \frac{1}{2} y_{ij} \right]$$

Equilibrium

As such, the problem for player 1, given y, is the following:

$$\max_x v_1(x,y) \text{ s.t. } \sum_i x_{ij} = 1 \ \forall j \text{ , and } \sum_j x_{ij} = 1 \ \forall i \tag{1}$$

where:

$$v_1(x,y) = \sum_i \sum_j r_i x_{ij} \left[\sum_{k>j} y_{ik} + \frac{1}{2} y_{ij} \right]$$

A Nash equilibrium is then a pair (x^*, y^*), such that x^* solves problem 1 given $y = y^*$, and y^* solves the equivalent problem for player 2 given $x = x^*$.

An equilibrium guess

The existence of a preemptive best reply (i.e. a cyclic shift in the search path) implies that, at least in the case with destructive ties, there won't be a pure strategy Nash equilibrium.

An equilibrium guess

The existence of a preemptive best reply (i.e. a cyclic shift in the search path) implies that, at least in the case with destructive ties, there won't be a pure strategy Nash equilibrium.

As such, we'd expect mixing to be a key part of any equilibrium. Suppose that at time step j, given y, player 1 mixes between i and i'. It must be that:

$$(1 - \sum_{k \leq j} y_{ik}) r_i = (1 - \sum_{k \leq j} y_{i'k}) r_{i'}$$

An equilibrium guess

The existence of a preemptive best reply (i.e. a cyclic shift in the search path) implies that, at least in the case with destructive ties, there won't be a pure strategy Nash equilibrium.

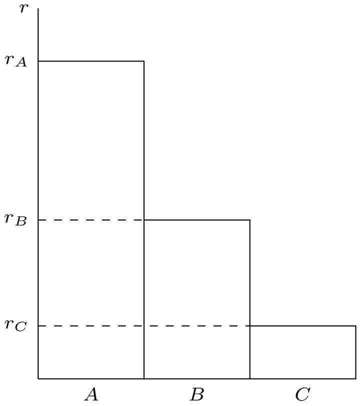
As such, we'd expect mixing to be a key part of any equilibrium. Suppose that at time step j, given y, player 1 mixes between i and i'. It must be that:

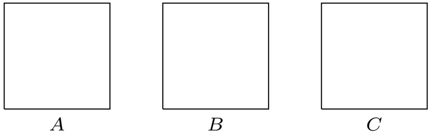
$$(1 - \sum_{k \leq j} y_{ik}) r_i = (1 - \sum_{k \leq j} y_{i'k}) r_{i'}$$

As such let's look for an equilibrium that "levels" these *posterior values* at each time step.

Building a leveling equilibrium

Take the following 3 box environment and start with $z_{ij} = 0 \; \forall i, j$.

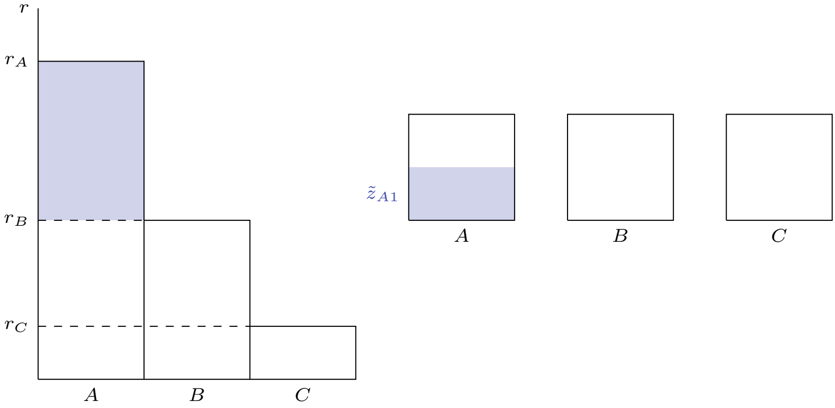




Building a leveling equilibrium

Define τ_k as the probability needed to reduce the posterior value of all boxes with an index equal or lower than k to r_{k+1}.

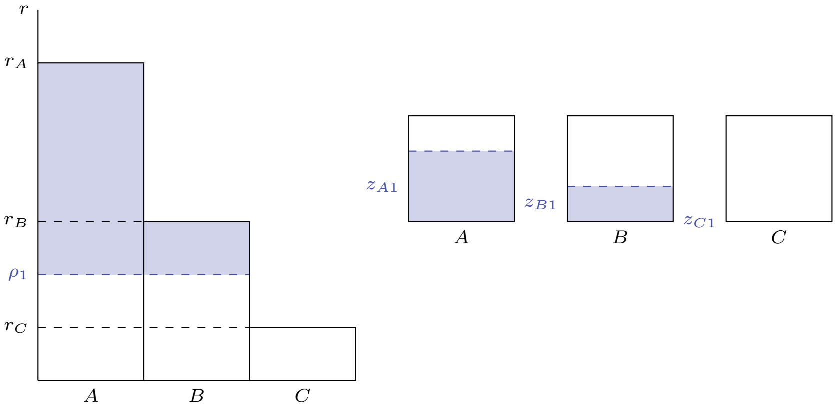
- So $\tau_A = \frac{r_A - r_B}{r_A}$.



Building a leveling equilibrium

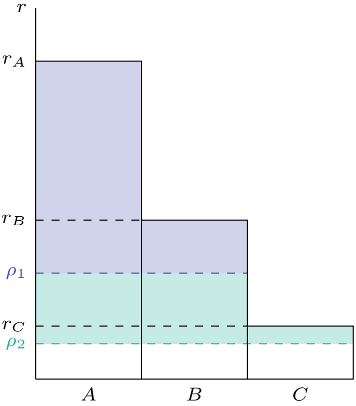
Define ρ_j as the leveled posterior value of the boxes at time-step j.

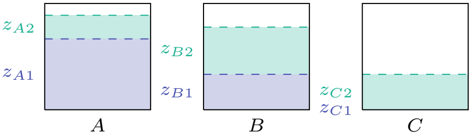
- Suppose for this example, $r_B > \rho_1 > r_C$.



Building a leveling equilibrium

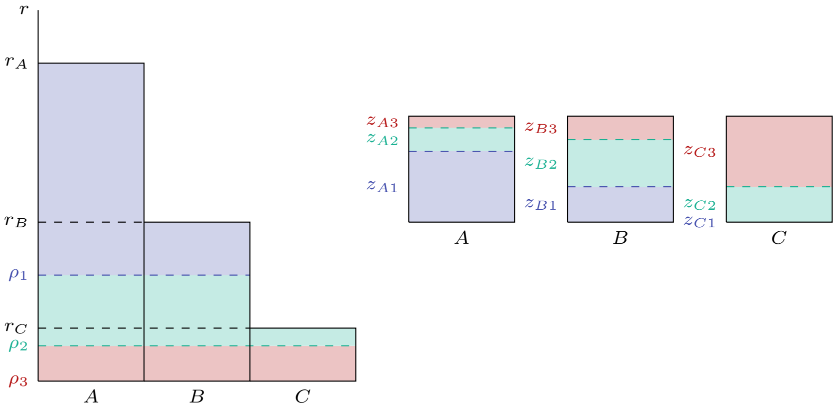
The second step proceeds as the last, with an additional 1 unit of probability to assign among the boxes.





Building a leveling equilibrium

This algorithm always terminates by levelling all posterior values to zero ($\rho_n = 0$).



The discrete leveling strategy

As such, we can write out the **discrete levelling strategy** as:

$$x_{ij} = \sum_{k=1}^{n} max\Big\{min\{\tau_k, j\} - max\{\tau_{k-1}, j-1\}, 0\Big\} \cdot w_{ik}$$

where:

$$\tau_k = \tau_{k-1} + (r_k - r_{k+1}) \sum_{\iota=1}^{k} \frac{1}{r_\iota} \quad \text{with} \quad \tau_0 = 0 \quad \text{and} \quad r_{n+1} = 0$$

and

$$w_{ik} = \begin{cases} \frac{\prod_{\iota=1,\iota\neq i}^{k} r_\iota}{\sum_{l=1}^{k} \prod_{\iota=1,\iota\neq l}^{k} r_\iota} & \text{if } i \leq k \\ 0 & \text{if } i > k \end{cases}$$

The discrete leveling strategy

As such, we can write out the **discrete levelling strategy** as:

$$x_{ij} = \sum_{k=1}^{n} max\Big\{min\{\tau_k, j\} - max\{\tau_{k-1}, j-1\}, 0\Big\} \cdot w_{ik}$$

where:

$$\tau_k = \tau_{k-1} + (r_k - r_{k+1}) \sum_{\iota=1}^{k} \frac{1}{r_\iota} \quad \text{with} \quad \tau_0 = 0 \quad \text{and} \quad r_{n+1} = 0$$

and

$$w_{ik} = \begin{cases} \frac{\prod_{\iota=1,\iota\neq i}^{k} r_\iota}{\sum_{l=1}^{k} \prod_{\iota=1,\iota\neq l}^{k} r_\iota} & \text{if } i \leq k \\ 0 & \text{if } i > k \end{cases}$$

Corollary

$\tau_1 < 1$, $\tau_j \leq j \ \forall j$ and $\tau_n = n$.

Results

Define the **discrete levelling profile** as the strategy profile where each player plays the discrete levelling strategy.

Theorem

The discrete levelling profile is the unique symmetric Nash equilibrium of the completely destructive ties game.

Results

Define the **discrete levelling profile** as the strategy profile where each player plays the discrete levelling strategy.

Theorem

The discrete levelling profile is the unique symmetric Nash equilibrium of the completely destructive ties game.

Theorem

The discrete levelling profile achieves the minimax value for each player.

Some comments

- This analysis extends easily to many players.
- Fershtman and Rubinstein (1997) study uniform games where, despite having even-split ties, the equilibrium is identical.
- Liu and Wong (2023) study continuous exploration where ties don't matter, and so the equilibrium corresponds to the discrete setup with destructive ties but **not** more generally.
- The applications an their associated policy choices are important, eg.,
  1. innovation and copyright,
  2. research and publication, and
  3. invention and patents.

References

Avis, David, Gabriel D. Rosenberg, Rahul Savani and Bernhard von Stengel. 2010. “Enumeration of Nash Equilibria for Two-Player Games.” *Economic Theory* 42(1):9–37.

Birkhoff, Garrett. 1946. “Three observations on linear algebra.” *Univ. Nac. Tucumán. Revista A.* 5:147–151.

Fershtman, Chaim and Ariel Rubinstein. 1997. “A Simple Model of Equilibrium in Search Procedures.” *Journal of Economic Theory* 72(2):432–441.

Immorlica, Nicole, Adam Tauman Kalai, Brendan Lucier, Ankur Moitra, Andrew Postlewaite and Moshe Tennenholtz. 2011. “Dueling Algorithms.” *CoRR* abs/1101.2883.

Liu, Qingmin and Yu Fu Wong. 2023. “Strategic Exploration: Preemption and Prioritization.” *Review of Economic Studies* 91:2425–2461.